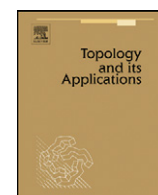




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## Topology and its Applications

[www.elsevier.com/locate/topol](http://www.elsevier.com/locate/topol)On the structure of the commutator subgroup of certain homeomorphism groups<sup>☆</sup>Ilona Michalik, Tomasz Rybicki<sup>\*</sup>

Faculty of Applied Mathematics, AGH University of Science and Technology, al. Mickiewicza 30, 30-059 Kraków, Poland

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## ABSTRACT

An important theorem of Ling states that if  $G$  is any factorizable non-fixing group of homeomorphisms of a paracompact space then its commutator subgroup  $[G, G]$  is perfect. This paper is devoted to further studies on the algebraic structure (e.g. uniform perfectness, uniform simplicity) of  $[G, G]$  and  $[\tilde{G}, \tilde{G}]$ , where  $\tilde{G}$  is the universal covering group of  $G$ . In particular, we prove that if  $G$  is a bounded factorizable non-fixing group of homeomorphisms then  $[G, G]$  is uniformly perfect (Corollary 3.4). The case of open manifolds is also investigated. Examples of homeomorphism groups illustrating the results are given.

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## 1. Introduction

Given groups  $G$  and  $H$ , by  $G \leq H$  (resp.  $G \triangleleft H$ ) we denote that  $G$  is a subgroup (resp. normal subgroup) of  $H$ . Throughout by  $\mathcal{H}(X)$  we denote the group of all homeomorphism of a topological space  $X$ . Let  $U$  be an open subset of  $X$  and let  $G$  be a subgroup of  $\mathcal{H}(X)$ . The symbol  $\mathcal{H}_U(X)$  (resp.  $G_U$ ) stands for the subgroup of elements of  $\mathcal{H}(X)$  (resp.  $G$ ) with support in  $U$ . For  $g \in \mathcal{H}(X)$  the support of  $g$ ,  $\text{supp}(g)$ , is the closure of  $\{x \in X : g(x) \neq x\}$ . Let  $\mathcal{H}_c(M)$  (resp.  $G_c$ ) denote the subgroup of  $\mathcal{H}(M)$  (resp.  $G$ ) of all its compactly supported elements.

**Definition 1.1.** Let  $\mathcal{U}$  be an open cover of  $X$ . A group of homeomorphisms  $G$  of a space  $X$  is called  $\mathcal{U}$ -factorizable if for every  $g \in G$  there are  $g_1, \dots, g_r \in G$  with  $g = g_1 \dots g_r$  and such that  $\text{supp}(g_i) \subset U_i$ ,  $i = 1, \dots, r$ , for some  $U_1, \dots, U_r \in \mathcal{U}$ .  $G$  is called factorizable if for every open cover  $\mathcal{U}$  of  $X$  it is  $\mathcal{U}$ -factorizable.

Next  $G$  is said to be non-fixing if  $G(x) \neq \{x\}$  for every  $x \in X$ , where  $G(x) := \{g(x) \mid g \in G\}$  is the orbit of  $G$  at  $x$ .

Given a group  $G$ , denote by  $[f, g] = fgf^{-1}g^{-1}$  the commutator of  $f, g \in G$ , and by  $[G, G]$  the commutator subgroup. Now the theorem of Ling can be formulated as follows.

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<sup>\*</sup> Corresponding author.

E-mail addresses: [imichali@wms.mat.agh.edu.pl](mailto:imichali@wms.mat.agh.edu.pl) (I. Michalik), [tomasz@uci.agh.edu.pl](mailto:tomasz@uci.agh.edu.pl) (T. Rybicki).

**Theorem 1.2.** ([15]) *Let  $X$  be a paracompact topological space and let  $G$  be a factorizable non-fixing group of homeomorphisms of  $X$ . Then the commutator subgroup  $[G, G]$  is perfect, that is  $[[G, G], [G, G]] = [G, G]$ .*

Recall that a group  $G$  is called *uniformly perfect* [5] if  $G$  is perfect (i.e.  $G = [G, G]$ ) and there exists a positive integer  $r$  such that any element of  $G$  can be expressed as a product of at most  $r$  commutators of elements of  $G$ . For  $g \in [G, G]$ ,  $g \neq e$ , the least  $r$  such that  $g$  is a product of  $r$  commutators is called the *commutator length* of  $g$  and is denoted by  $\text{cl}_G(g)$ . By definition we put  $\text{cl}_G(e) = 0$ .

Throughout we adopt the following notation. Let  $M$  be a paracompact manifold of class  $C^r$ , where  $r = 0, 1, \dots, \infty$ . Then  $\mathcal{D}^r(M)$  (resp.  $\mathcal{D}_c^r(M)$ ) denotes the group of all  $C^r$ -diffeomorphisms of  $M$  which can be joined with the identity by a (resp. compactly supported)  $C^r$ -isotopy. For simplicity by  $C^0$ -diffeomorphism we mean a homeomorphism.

Observe that in view of recent results (Burago, Ivanov and Polterovich [5], Tsuboi [29]) the diffeomorphism groups  $\mathcal{D}_c^\infty(M)$  are uniformly perfect for most types of manifolds  $M$ , though some open problems are left.

Our first aim is to prove the following generalization of Theorem 1.2.

**Theorem 1.3.** *Let  $X$  be a paracompact topological space and let  $G$  be a factorizable non-fixing group of homeomorphisms of  $X$ . Assume that  $\text{cl}_G$  is bounded on  $[G, G]$  and that  $G$  is bounded with respect to all fragmentation norms  $\text{frag}^{\mathcal{U}}$  (cf. Section 2), where  $\mathcal{U}$  runs over all open covers of  $X$ . Then the commutator subgroup  $[G, G]$  is uniformly perfect.*

The proof of Theorem 1.3 and further results concerning the uniform perfectness of  $[G, G]$  will be given in Section 3.

Ling's theorem (Theorem 1.2) constitutes an essential amelioration of the Epstein simplicity theorem [8] at least in two aspects. First, contrary to [8], it provides an algebraic information on nontransitive homeomorphism groups. Second, it enables to strengthen the theorem of Epstein itself. We will recall Epstein's theorem and Ling's improvement of it in Section 4. Also in Section 4 we formulate conditions which ensure the uniform simplicity of  $[G, G]$  (Theorem 4.3).

As usual  $\tilde{G}$  stands for the universal covering group of  $G$ . In Section 5 we will prove the following:

**Theorem 1.4.** *Suppose that  $G \leq \mathcal{H}(X)$  is isotopically factorizable (Definition 5.2) and that  $G_0$ , the identity component of  $G$ , is non-fixing. Then the commutator group  $[G, G]$  is perfect.*

In Section 6 we will consider the case of a noncompact manifold  $M$  such that  $M$  is the interior of a compact manifold  $\bar{M}$ , and groups of homeomorphisms on  $M$  with no restriction on support. Consequently such groups are not factorizable in the usual way but only in a wider sense (Definition 6.1). It is surprising that for a large class of homeomorphism or diffeomorphism groups of an open manifold the assertions of Theorems 1.2 and 1.3 still hold (see Theorems 6.9 and 6.10).

In the final section we will present some examples and open problems which are of interest in the context of the above results.

## 2. Conjugation-invariant norms

The notion of the conjugation-invariant norm is a basic tool in studies on the structure of groups. Let  $G$  be a group. A *conjugation-invariant norm* (or *norm* for short) on  $G$  is a function  $\nu : G \rightarrow [0, \infty)$  which satisfies the following conditions. For any  $g, h \in G$ :

- (1)  $\nu(g) > 0$  if and only if  $g \neq e$ ;
- (2)  $\nu(g^{-1}) = \nu(g)$ ;
- (3)  $\nu(gh) \leq \nu(g) + \nu(h)$ ;
- (4)  $\nu(hgh^{-1}) = \nu(g)$ .

Recall that a group is called *bounded* if it is bounded with respect to any bi-invariant metric. It is easily seen that  $G$  is bounded if and only if any conjugation-invariant norm on  $G$  is bounded.

Observe that the commutator length  $\text{cl}_G$  is a conjugation-invariant norm on  $[G, G]$ . In particular, if  $G$  is a perfect group then  $\text{cl}_G$  is a conjugation-invariant norm on  $G$ . For any perfect group  $G$  denote by  $\text{cld}_G$  the commutator length diameter of  $G$ , i.e.  $\text{cld}_G := \sup_{g \in G} \text{cl}_G(g)$ . Then  $G$  is uniformly perfect iff  $\text{cld}_G < \infty$ .

Assume now that  $G \leq \mathcal{H}(X)$  is  $\mathcal{U}$ -factorizable (Definition 1.1), and that  $\mathcal{U}$  is a  $G$ -invariant open cover of  $X$ . The latter means that  $g(U) \in \mathcal{U}$  for all  $g \in G$  and  $U \in \mathcal{U}$ . Then we may introduce the following conjugation-invariant norm  $\text{frag}^{\mathcal{U}}$  on  $G$ . Namely, for  $g \in G$ ,  $g \neq \text{id}$ , we define  $\text{frag}^{\mathcal{U}}(g)$  to be the least integer  $\rho > 0$  such that  $g = g_1 \dots g_\rho$  with  $\text{supp}(g_i) \subset U_i$  for some  $U_i \in \mathcal{U}$ , where  $i = 1, \dots, \rho$ . By definition  $\text{frag}^{\mathcal{U}}(\text{id}) = 0$ .

Define  $\text{frag}_G^{\mathcal{U}} := \sup_{g \in G} \text{frag}^{\mathcal{U}}(g)$ , the diameter of  $G$  in  $\text{frag}^{\mathcal{U}}$ . Consequently,  $\text{frag}^{\mathcal{U}}$  is bounded iff  $\text{frag}_G^{\mathcal{U}} < \infty$ .

Observe that  $\text{frag}^{(X)}$  is the trivial norm on  $G$ , i.e. equal to 1 for all  $g \in G \setminus \{\text{id}\}$ . Observe as well that  $\text{frag}^{\mathcal{V}} \geq \text{frag}^{\mathcal{U}}$  provided  $\mathcal{V}$  is finer than  $\mathcal{U}$ .

The significance of  $\text{frag}^{\mathcal{U}}$  consists in the following version of Proposition 1.15 in [5].

**Proposition 2.1.** Let  $M$  be a  $C^r$ -manifold,  $r = 0, 1, \dots, \infty$ . Then  $\mathcal{D}_c^r(M)$  is bounded if and only if  $\mathcal{D}_c^r(M)$  is bounded with respect to  $\text{frag}^{\mathcal{U}}$ , where  $\mathcal{U}$  is some cover by embedded open balls.

Indeed, it is a consequence of Theorem 1.18 in [5] stating that for a portable manifold  $M$  the group  $\mathcal{D}_c^r(M)$  is bounded, and the fact that  $\mathbb{R}^n$  is portable.

### 3. Uniform perfectness of $[G, G]$

In Theorems 3.5 and 3.8 below we also need stronger notions than that of non-fixing group (Definition 1.1).

**Definition 3.1.** Let  $\mathcal{U}$  be an open cover of  $X$ ,  $G \leq \mathcal{H}(X)$  and let  $r \in \mathbb{N}$ .

- (1)  $G$  is called 1-non-fixing if for any  $x \in X$  there are  $f, g \in G$  such that  $[f, g](x) \neq x$ .
- (2)  $G$  is said to be  $\mathcal{U}$ -moving if for every  $U \in \mathcal{U}$  then there is  $g \in G$  such that  $g(U) \cap U = \emptyset$ .
- (3)  $G$  is said to be  $r$ - $\mathcal{U}$ -moving if for any  $U \in \mathcal{U}$  there are  $2r$  elements of  $G$  (possibly = id), say  $f_1, \dots, f_r, g_1, \dots, g_r$ , such that the sets  $U$  and  $([f_r, g_r] \dots [f_1, g_1])(U)$  are disjoint.
- (4)  $G$  is said to be strongly  $\mathcal{U}$ -moving if for every  $U, V \in \mathcal{U}$  there is  $g \in G$  such that  $g(U) \cap (U \cup V) = \emptyset$ .
- (5)  $G$  is called locally moving if for any open set  $U \subset X$  and  $x \in U$  there is  $g \in G_U$  such that  $g(x) \neq x$ .

Of course, if  $G$  is either  $r$ -non-fixing, or  $\mathcal{U}$ -moving, or locally moving then it is non-fixing. Likewise, if  $G$  is  $r$ - $\mathcal{U}$ -moving then it is  $s$ - $\mathcal{U}$ -moving for  $r < s$  and  $\mathcal{U}$ -moving. Notice that if  $\mathcal{V}$  is finer than  $\mathcal{U}$  and  $G$  is (resp. strongly)  $\mathcal{U}$ -moving then  $G$  is (resp. strongly)  $\mathcal{V}$ -moving.

**Proposition 3.2.** Let  $X$  be paracompact and let  $G \leq \mathcal{H}(X)$ .

- (1) If  $G$  is non-fixing and factorizable (Definition 1.1) then  $G$  is locally moving.
- (2) If  $G$  is locally moving then so is  $[G, G]$ .
- (3) If  $G$  is non-fixing and factorizable then  $[G, G]$  is 1-non-fixing (Definition 3.1(1)).

**Proof.** (1) Let  $x \in U$  and  $g \in G$  such that  $g(x) = y \neq x$ . Choose  $\mathcal{U} = \{U_1, U_2\}$ , where  $x \in U_1 \setminus U_2$ ,  $y \in U_2 \setminus U_1$ ,  $U_1 \subset U$  and  $X = U_1 \cup U_2$ . By assumption we may write  $g = g_r \dots g_1$ , where all  $g_i$  are supported in elements of  $\mathcal{U}$ . Let  $s := \min\{i \in \{1, \dots, r\} : \text{supp}(g_i) \subset U_1 \text{ and } g_i(x) \neq x\}$ . Then  $g_s \in G_U$  satisfies  $g_s(x) \neq x$ .

(2) Let  $x \in U$ . There is  $g \in G_U$  with  $g(x) \neq x$ . Take an open  $V$  such that  $x \in V \subset U$  and  $g(x) \notin V$ . Choose  $f \in G_V$  with  $f(x) \neq x$ . It follows that  $f(g(x)) = g(x) \neq g(f(x))$  and, therefore,  $[f^{-1}, g^{-1}](x) \neq x$ . (3) follows from (1) and the proof of (2).  $\square$

The following property of paracompact spaces is well known.

**Lemma 3.3.** If  $X$  is a paracompact space and  $\mathcal{U}$  is an open cover of  $X$ , then there exists an open cover  $\mathcal{V}$  star finer than  $\mathcal{U}$ , that is for all  $V \in \mathcal{V}$  there is  $U \in \mathcal{U}$  such that  $\text{star}^{\mathcal{V}}(V) \subset U$ . Here  $\text{star}^{\mathcal{V}}(V) := \bigcup\{V' \in \mathcal{V} : V' \cap V \neq \emptyset\}$ . In particular, for all  $V_1, V_2 \in \mathcal{V}$  with  $V_1 \cap V_2 \neq \emptyset$  there is  $U \in \mathcal{U}$  such that  $V_1 \cup V_2 \subset U$ .

If  $\mathcal{V}$  and  $\mathcal{U}$  are as in Lemma 3.3 then we will write  $\mathcal{V} \prec \mathcal{U}$ .

For an open cover  $\mathcal{U}$  let  $\mathcal{U}^G := \{g(U) : g \in G \text{ and } U \in \mathcal{U}\}$ .

**Proof of Theorem 1.3.** In view of Proposition 3.2 and the assumption, for any  $x \in X$  there is  $f, g \in [G, G]$  such that  $[f, g](x) \neq x$ . It follows the existence of an open cover  $\mathcal{U}$  such that for any  $U \in \mathcal{U}$  there are  $f, g \in [G, G]$  such that  $[f, g](U) \cap U = \emptyset$ . Hence we have also that for any  $U \in \mathcal{U}^G$  there is  $f, g \in [G, G]$  such that  $[f, g](U) \cap U = \emptyset$ . In fact, if  $N \triangleleft G$  and  $U \in \mathcal{U}$  such that  $n(U) \cap U = \emptyset$  for some  $n \in N$ , then for  $g \in G$  we get  $(\bar{n}g)(U) \cap g(U) = \emptyset$ , where  $\bar{n} = gng^{-1} \in N$ .

Due to Lemma 3.3 we can find  $\mathcal{V}$  such that  $\mathcal{V} \prec \mathcal{U}$ . We denote

$$G^{\mathcal{U}} = \prod_{U \in \mathcal{U}^G} [G_U, G_U].$$

Assume that  $G$  is  $\mathcal{V}$ -factorizable and  $\text{fragd}_G^{\mathcal{V}} = \rho$ . First we show that  $[G, G] \subset G^{\mathcal{U}}$  and that any  $[g_1, g_2] \in [G, G]$  can be expressed as a product of at most  $\rho^2$  elements of  $G^{\mathcal{U}}$  of the form  $[h_1, h_2]$ , where  $h_1, h_2 \in G_U$  for some  $U$ . In fact, it is an immediate consequence of the following commutator formulae for all  $f, g, h \in G$

$$[fg, h] = f[g, h]f^{-1}[f, h], \quad [f, gh] = [f, g]g[f, h]g^{-1}, \quad (3.1)$$

and the fact that  $\mathcal{V} \prec \mathcal{U}$ . Now if  $\text{cld}_G = d$ , then every element of  $[G, G]$  is a product of at most  $d\rho^2$  elements of  $G^{\mathcal{U}}$  of the form  $[h_1, h_2]$ , where  $h_1, h_2 \in G_U$  for some  $U$ .

Next, fix arbitrarily  $U \in \mathcal{U}$ . We have to show that for every  $f, g \in G_U$  the bracket  $[f, g]$  can be represented as a product of four commutators of elements of  $[G, G]$ . By assumption on  $\mathcal{U}^G$ , there are  $h_1, h_2 \in [G, G]$  such that  $h(U) \cap U = \emptyset$ , where  $h = [h_1, h_2]$ . It follows that  $[hfh^{-1}, g] = \text{id}$ . Therefore,  $[[f, h], g] = [f, g]$ . Observe that if  $k_1$  is a product of  $s$  commutators of elements of  $[G, G]$  and  $k_2 \in G$ , then  $[k_1, k_2]$  is a product of  $2s$  commutators of elements of  $[G, G]$ . It follows that indeed  $[[f, h], g]$  is a product of four commutators of elements of  $[G, G]$ . Thus any element of  $[G, G]$  is a product of at most  $4d\rho^2$  commutators of elements of  $[G, G]$ .  $\square$

**Corollary 3.4.** *Let  $X$  be a paracompact space and let  $G \leq \mathcal{H}(X)$  be a bounded, factorizable and non-fixing group. Then the commutator subgroup  $[G, G]$  is uniformly perfect.*

**Proof.** The only thing we need is that  $\text{cl}_G$  should be bounded (on  $[G, G]$ ), and this fact is a consequence of Proposition 4.1 in [5] stating that any group  $G$  is unbounded provided that the commutator length on  $[G, G]$  is unbounded.  $\square$

A more refined version of Theorem 1.3 is the following:

**Theorem 3.5.** *Let  $X$  be a paracompact topological space, let  $G \leq \mathcal{H}(X)$  with  $\text{cl}_G$  bounded (as the norm on  $[G, G]$ ) and let  $\mathcal{U}$  be a  $G$ -invariant open cover of  $X$  such that*

- (1)  $G$  is strongly  $\mathcal{U}$ -moving (Definition 3.1(4)), and
- (2) *there is an open cover  $\mathcal{V}$  satisfying  $\mathcal{V} \prec \mathcal{U}$  such that  $G$  is  $\mathcal{V}$ -factorizable and  $G$  is bounded with respect to the fragmentation norm  $\text{frag}^{\mathcal{V}}$ .*

*Then the commutator subgroup  $[G, G]$  is uniformly perfect. Furthermore, if  $\text{frag}_G^{\mathcal{V}} = \rho$  and  $\text{cld}_G = d$  then  $\text{cld}_{[G, G]} \leq d\rho^2$ .*

**Proof.** Let  $\mathcal{U}$  and  $\mathcal{V}$  satisfy the assumption. We denote

$$G^{\mathcal{U}} = \prod_{U \in \mathcal{U}} [G_U, G_U].$$

As in the proof of Theorem 1.3, first we show, due to (3.1) and  $G$ -invariance of  $\mathcal{U}$ , that  $[G, G] \subset G^{\mathcal{U}}$  and that any  $[f, g] \in [G, G]$  can be written as a product of at most  $\rho^2$  elements of  $G^{\mathcal{U}}$  of the form  $[h_1, h_2]$ , where  $h_1, h_2 \in G_U$  for some  $U$ . This implies that every element of  $[G, G]$  is a product of at most  $d\rho^2$  elements of  $G^{\mathcal{U}}$  of the form  $[h_1, h_2]$ , where  $h_1, h_2 \in G_U$  for some  $U$ .

For  $U \in \mathcal{U}$  we will show that for every  $f, g \in G_U$  the bracket  $[f, g]$  is a commutator of two elements of  $[G, G]$ . By assumption and Definition 3.1(4), there is  $h \in G$  such that  $h(U) \cap U = \emptyset$ . It follows that  $[hfh^{-1}, g] = \text{id}$ . Next, for  $U, h(U) \in \mathcal{U}$  there is  $k \in G$  such that  $k(U) \cap (U \cup h(U)) = \emptyset$ . Consequently,  $[f, kg^{-1}k^{-1}] = \text{id}$  and  $[hf^{-1}h^{-1}, kg^{-1}k^{-1}] = \text{id}$ . Therefore,  $[f, g] = [[f, h], [g, k]]$ , that is  $[f, g]$  is a commutator of elements of  $[G, G]$ . Thus  $[G, G]$  is uniformly perfect and  $\text{cld}_{[G, G]} \leq d\rho^2$ , as required.  $\square$

From the proof of Theorem 3.5 we get:

**Corollary 3.6.** *If  $\mathcal{U}$  is a  $G$ -invariant open cover of  $X$  such that  $G$  is strongly  $\mathcal{U}$ -moving and  $\mathcal{V}$ -factorizable for some open cover  $\mathcal{V}$  satisfying  $\mathcal{V} \prec \mathcal{U}$  then  $[G, G]$  is perfect.*

**Proposition 3.7.**

- (1) *Let  $G$  be  $\mathcal{U}$ -moving. Assume that  $\mathcal{V}$  is a  $G$ -invariant open cover such that  $\mathcal{V} \prec \mathcal{U}$ ,  $G$  is  $\mathcal{V}$ -factorizable and  $\text{frag}_G^{\mathcal{V}} = \rho$ . Then  $G$  is  $\rho$ - $\mathcal{V}$ -moving.*
- (2) *Let  $\mathcal{U}, \mathcal{W}$  and  $\mathcal{T}$  be such that  $\mathcal{T} \prec \mathcal{W} \prec \mathcal{V} \prec \mathcal{U}$ , and  $\mathcal{V}, \mathcal{W}$  and  $\mathcal{T}$  are  $G$ -invariant. If  $G$  is  $\mathcal{U}$ -moving and  $\mathcal{T}$ -factorizable with  $\text{frag}_G^{\mathcal{T}} = \rho$ , then  $[G, G]$  is  $\rho^2$ - $\mathcal{W}$ -moving.*

**Proof.** (1) Suppose that  $\mathcal{V} \prec \mathcal{U}$  and let  $V \in \mathcal{V}$ . Then there is  $g \in G$  such that  $g(V) \cap V = \emptyset$ . By assumption there exist  $V_1, \dots, V_\rho \in \mathcal{V}$  and  $g_1, \dots, g_\rho \in G$  such that  $g = g_\rho \dots g_1$  and  $\text{supp}(g_i) \subset V_i$ ,  $i = 1, \dots, \rho$  (possibly  $g_i = \text{id}$ ).

Let us consider two cases: (a)  $g_1(V) \cap V = \emptyset$  and (b)  $g_1(V) \cap V \neq \emptyset$ . In case (a) we have  $g_1(V) \cup V \subset \text{supp}(g_1) \subset U \in \mathcal{U}$ . Choose  $f_1 \in G$  such that  $f_1(U) \cap U = \emptyset$ . Then  $[g_1, f_1](V) = g_1(V)$  and we are done. In case (b)  $V \cup g_1(V) \subset U_1 \in \mathcal{U}$  such that  $f_1(U_1) \cap U_1 = \emptyset$  for some  $f_1 \in G$ . Again  $[g_1, f_1](V) = g_1(V)$ . Now we continue as before. In case  $(g_2 g_1)(V) \cap g_1(V) = \emptyset$  we get  $V \cap \bar{g}_2(V) = \emptyset$ , where  $\bar{g}_2 = g_1^{-1} g_2^{-1} g_1$ , and we are done as in (a). Otherwise,  $(g_2 g_1)(V) \cup g_1(V) \subset U_2 \in \mathcal{U}$  such that  $f_2(U_2) \cap U_2 = \emptyset$  for some  $f_2 \in G$ . Therefore,  $[g_2, f_2](g_1(V)) = (g_2 g_1)(V)$ . Proceeding by induction we get

$$([g_\rho, f_\rho] \dots [g_1, f_1])(V) = (g_\rho \dots g_1)(V) = g(V),$$

and the claim follows.

(2) It follows from the hypotheses that  $G$  is  $\mathcal{V}$ -factorizable and  $\text{fragd}_G^{\mathcal{V}} \leq \rho$ . Moreover, as in the proof of Theorem 1.3 we get that  $[G, G]$  is  $\mathcal{W}$ -factorizable and  $\text{fragd}_{[G, G]}^{\mathcal{W}} \leq \rho^2$ . Hence by (1)  $G$  is  $\rho$ - $\mathcal{V}$ -moving. In particular  $[G, G]$  is  $\mathcal{V}$ -moving. Then again (1) implies that  $[G, G]$  is  $\rho^2$ - $\mathcal{W}$ -moving.  $\square$

In the following version of Theorem 1.3 we avoid the assumption that  $G$  is strongly  $\mathcal{U}$ -moving.

**Theorem 3.8.** *Let  $X$  be a paracompact topological space, let  $G \leq \mathcal{H}(X)$  with  $\text{cl}_G$  bounded, and let  $\mathcal{U}$  be an open cover of  $X$  such that*

- (1)  $G$  is  $\mathcal{U}$ -moving, and
- (2) *there are  $G$ -invariant open covers  $\mathcal{V}$ ,  $\mathcal{W}$ , and  $\mathcal{T}$  fulfilling the relation  $\mathcal{T} \prec \mathcal{W} \prec \mathcal{V} \prec \mathcal{U}$ , and such that  $G$  is  $\mathcal{T}$ -factorizable and it is bounded with respect to  $\text{frag}^{\mathcal{T}}$ .*

*Then  $[G, G]$  is uniformly perfect and  $\text{cld}_{[G, G]} \leq 4d\rho^4$  provided  $\text{fragd}_G^{\mathcal{T}} = \rho$  and  $\text{cld}_G = d$ .*

**Proof.** Let  $\text{fragd}_G^{\mathcal{T}} = \rho$ . Then a fortiori  $\text{fragd}_G^{\mathcal{W}} \leq \rho$ . In view of Proposition 3.7,  $[G, G]$  is  $\rho^2$ - $\mathcal{W}$ -moving.

Let  $[f, g] \in [G, G]$ . By applying for  $\mathcal{T} \prec \mathcal{W}$  the same reasoning as in the proof of Theorem 1.3 for  $\mathcal{V} \prec \mathcal{U}$ ,  $[f, g]$  can be written as a product of at most  $\rho^2$  elements from  $G^{\mathcal{W}} = \prod_{W \in \mathcal{W}} [G_W, G_W]$  of the form  $[h_1, h_2]$ , where  $h_1, h_2 \in G_W$  for some  $W \in \mathcal{W}$ . Consequently, every element of  $[G, G]$  can be expressed as a product of at most  $d\rho^2$  elements of  $G^{\mathcal{W}}$  of the form  $[h_1, h_2]$ , where  $h_1, h_2 \in G_W$  for some  $W \in \mathcal{W}$ .

Now take arbitrarily  $W \in \mathcal{W}$  and  $f, g \in G_W$ . Since  $[G, G]$  is  $\rho^2$ - $\mathcal{W}$ -moving, there are  $h_1, \dots, h_{\rho^2}, h'_1, \dots, h'_{\rho^2} \in [G, G]$  such that for  $h = [h_1, h'_1] \dots [h_{\rho^2}, h'_{\rho^2}]$  we have  $h(W) \cap W = \emptyset$  and, consequently,  $[[f, h], g] = [f, g]$ . It is easily seen that  $[[f, h], g]$  is a product of  $4\rho^2$  commutators of elements of  $[G, G]$ . Thus any element of  $[G, G]$  is a product of at most  $4d\rho^4$  commutators of elements of  $[G, G]$ .  $\square$

As a consequence of the above proof we have:

**Corollary 3.9.** *If  $G$  is  $\mathcal{U}$ -moving and  $\mathcal{T}$ -factorizable for some  $G$ -invariant open covers  $\mathcal{V}$ ,  $\mathcal{W}$ , and  $\mathcal{T}$  such that  $\mathcal{T} \prec \mathcal{W} \prec \mathcal{V} \prec \mathcal{U}$ , then  $[G, G]$  is perfect.*

#### 4. Simplicity and uniform simplicity of $[G, G]$

Let us recall Epstein's theorem.

**Theorem 4.1.** ([8]) *Let  $X$  be a paracompact space, let  $G$  be a group of homeomorphisms of  $X$  and let  $\mathcal{B}$  be a basis of open sets of  $X$  satisfying the following axioms:*

Axiom 1. *If  $U \in \mathcal{B}$  and  $g \in G$ , then  $g(U) \in \mathcal{B}$ .*

Axiom 2.  *$G$  acts transitively on  $\mathcal{B}$  (i.e.  $\forall U, V \in \mathcal{B} \exists g \in G: g(U) = V$ ).*

Axiom 3. *Let  $g \in G$ ,  $U \in \mathcal{B}$  and let  $\mathcal{U} \subset \mathcal{B}$  be a cover of  $X$ . Then there exist an integer  $n$ , elements  $g_1, \dots, g_n \in G$  and  $V_1, \dots, V_n \in \mathcal{U}$  such that  $g = g_n g_{n-1} \dots g_1$ ,  $\text{supp}(g_i) \subset V_i$  and*

$$\text{supp}(g_i) \cup (g_{i-1} \dots g_1(\bar{U})) \neq X \quad \text{for } 1 \leq i \leq n.$$

*Then  $[G, G]$ , the commutator subgroup of  $G$ , is simple.*

It is worth noting that Theorem 4.1 was an indispensable ingredient in the proofs of celebrated simplicity theorems on diffeomorphism groups and their generalizations (cf. [27,16,3,4,11,18]).

We say that  $G \leq \mathcal{H}(X)$  acts *transitively inclusively* (cf. [15]) on a topological basis  $\mathcal{B}$  if for all  $U, V \in \mathcal{B}$  there is  $g \in G$  such that  $g(U) \subset V$ . It is not difficult to derive from Theorem 1.2 the following amelioration of Theorem 4.1, see [15].

**Theorem 4.2.** ([15]) *Let  $X$  be a paracompact space, let  $G \leq \mathcal{H}(X)$  and let  $\mathcal{B}$  be a basis of open sets of  $X$  satisfying the following axioms:*

Axiom 1.  *$G$  acts transitively inclusively on  $\mathcal{B}$ .*

Axiom 2.  *$G$  is  $\mathcal{U}$ -factorizable (Definition 1.1) for all covers  $\mathcal{U} \subset \mathcal{B}$ .*

*Then  $[G, G]$  is a simple group.*

Now we wish to provide conditions ensuring that the commutator group of a homeomorphism group is uniformly simple. Recall that a group  $G$  is called *uniformly simple* if there is  $d > 0$  such that for all  $f, g \in G$  with  $f \neq e$  we have

$g = h_1 \tilde{f} h_1^{-1} \dots h_s \tilde{f} h_s^{-1}$ , with  $\tilde{f} = f$  or  $\tilde{f} = f^{-1}$ , where  $s \leq d$  and  $h_1, \dots, h_s \in G$ . Given a uniformly simple group  $G$ , denote by  $\text{usd}_G$  the least  $d$  as above.

Note that recently Tsuboi [30] showed that  $\mathcal{D}_c^\infty(M)$  is uniformly simple for many types of manifolds  $M$ . However, for some types of  $M$  the problem is still unsolved.

**Theorem 4.3.** Let  $\mathcal{B}$  be a topological basis of  $X$ . Suppose that  $G \leq \mathcal{H}(X)$  satisfies the following conditions:

- (1)  $\text{cl}_G$  is bounded;
- (2)  $G$  acts transitively inclusively on  $\mathcal{B}$ ;
- (3) there is an open cover  $\mathcal{U} \prec \mathcal{B}$  such that  $G$  is  $\mathcal{U}$ -factorizable and  $G$  is bounded w.r.t. the fragmentation norm  $\text{frag}_G^\mathcal{U}$ .

Then the group  $[G, G]$  is uniformly simple. Moreover, if  $\text{cld}_G = d$  and  $\text{frag}_G^\mathcal{U} = \rho$  then  $\text{usd}_G \leq 4d\rho^2$ .

**Proof.** In view of Theorem 4.2,  $[G, G]$  is simple. Let  $f, g \in [G, G]$  such that  $f \neq e$ . There is  $x \in X$  with  $f(x) \neq x$  and  $B \in \mathcal{B}$  satisfying  $f(B) \cap B = \emptyset$ .

First we assume that  $g = [g_1, g_2] \in [G, G]$ . Then, if  $\text{frag}_G^\mathcal{U} = \rho$  then  $g$  can be expressed as a product of at most  $\rho^2$  elements of  $G^\mathcal{B} = \prod_{U \in \mathcal{B}^G} [G_U, G_U]$  of the form  $[h_1, h_2]$ , where  $h_1, h_2 \in G_U$  for some  $U \in \mathcal{B}^G$ . Here  $\mathcal{B}^G = \{g(U) \mid g \in G, U \in \mathcal{B}\}$ . In fact, we repeat the use of (3.1) as in the proof of Theorem 3.1. Now if  $\text{cld}_G = d$ , then every  $g \in [G, G]$  is a product of at most  $d\rho^2$  elements of  $G^\mathcal{B}$  of the form  $[h_1, h_2]$ , where  $h_1, h_2 \in G_U$  for some  $U \in \mathcal{B}^G$ .

Since  $G$  acts transitively inclusively on  $\mathcal{B}$  (and, consequently, on  $\mathcal{B}^G$ ), any  $[h_1, h_2]$  as above is conjugate to  $[k_1, k_2]$  with  $k_1, k_2 \in G_B$ . Then  $[k_1, k_2] = [[k_1, f], k_2]$ . Hence  $[k_1, k_2]$  is a product of four conjugates of  $f$  and  $f^{-1}$ . It follows that  $g$  is a product of at most  $4d\rho^2$  conjugates of  $f$  and  $f^{-1}$ , as claimed.  $\square$

**Corollary 4.4.** If  $G \leq \mathcal{H}(X)$  is factorizable and bounded, and  $G$  acts transitively inclusively on some basis  $\mathcal{B}$  of  $X$ , then  $[G, G]$  is uniformly simple.

In fact, in view of Proposition 1.4 in [5]  $[G, G]$  is then bounded in  $\text{cl}_G$ , and the remaining hypotheses of Theorem 4.3 are fulfilled too.

## 5. Perfectness of $(\tilde{G}, \tilde{G})$

Let  $G$  be a topological group. By  $\mathcal{P}G$  we will denote the totality of paths (or isotopies)  $\gamma: I \rightarrow G$  with  $\gamma(0) = e$  (where  $I = [0, 1]$ ). Then  $\mathcal{P}G$  endowed with the pointwise multiplication is a topological group. Next,  $\tilde{G}$  will stand for the universal covering group of  $G_0$ , the path connected identity component of  $G$ , that is  $\tilde{G} = \mathcal{P}G/\sim$ , where  $\sim$  denotes the relation of the homotopy rel. endpoints, provided  $G$  is locally path connected and semi-locally simply connected.

We introduce the following two operations on the space of paths  $\mathcal{P}G$ . Let  $\mathcal{P}^*G = \{\gamma \in \mathcal{P}G: \gamma(t) = e \text{ for } t \in [0, \frac{1}{2}]\}$ . For all  $\gamma \in \mathcal{P}G$  we define  $\gamma^*$  as follows:

$$\gamma^*(t) = \begin{cases} e & \text{for } t \in [0, \frac{1}{2}], \\ \gamma(2t-1) & \text{for } t \in [\frac{1}{2}, 1]. \end{cases}$$

Then  $\gamma^* \in \mathcal{P}^*G$  and the subgroup  $\mathcal{P}^*G$  is the image of  $\mathcal{P}G$  by the mapping  $\star: \gamma \mapsto \gamma^*$ . The elements of  $\mathcal{P}^*G$  are said to be special paths in  $G$ . Clearly, the group of special paths is preserved by conjugations, i.e. for each  $g \in \mathcal{P}G$  we have  $\text{conj}_g(\mathcal{P}^*G) \subset \mathcal{P}^*G$  for every  $g \in \mathcal{P}G$ , where  $\text{conj}_g(h) = ghg^{-1}$ ,  $h \in \mathcal{P}G$ .

Next, let  $\mathcal{P}^\square G = \{\gamma \in \mathcal{P}G: \gamma(t) = \gamma(1) \text{ for } t \in [\frac{1}{2}, 1]\}$ . For all  $\gamma \in \mathcal{P}G$  we define  $\gamma^\square$  by:

$$\gamma^\square(t) = \begin{cases} \gamma(2t) & \text{for } t \in [0, \frac{1}{2}], \\ \gamma(1) & \text{for } t \in [\frac{1}{2}, 1]. \end{cases}$$

As before  $\gamma^\square \in \mathcal{P}^\square G$  and the subgroup  $\mathcal{P}^\square G$  coincides with the image of  $\mathcal{P}G$  by the mapping  $\square: \gamma \mapsto \gamma^\square$ .

**Lemma 5.1.** For any  $\gamma \in \mathcal{P}G$  we have  $\gamma \sim \gamma^*$  and  $\gamma \sim \gamma^\square$ .

**Proof.** We have to find a homotopy  $\Gamma$  rel. endpoints between  $\gamma$  and  $\gamma^*$ . For all  $s \in I$  define  $\Gamma$  as follows:

$$\Gamma(t, s) = \begin{cases} e & \text{for } t \in [0, \frac{s}{2}], \\ \gamma(\frac{2t-s}{2-s}) & \text{for } t \in [\frac{s}{2}, 1]. \end{cases}$$

It is easy to check that such  $\Gamma$  fulfills all the requirements. Analogously the second claim follows.  $\square$

After these prerequisites let us return to homeomorphism groups.

Let  $X$  be a paracompact space and let  $G \leq \mathcal{H}(X)$ . Here  $\mathcal{H}(X)$  is endowed with the compact-open topology and  $G$  with the induced topology. If  $f \in \mathcal{P}G$  then we define  $\text{supp}(f) := \bigcup_{t \in [0,1]} \text{supp}(f_t)$ . By  $G_0$  we define the subgroup of all  $g \in G$  such that there is  $f \in \mathcal{P}G$  such that  $f_1 = g$ .  $G_0$  is called the *identity component* of  $G$ . Clearly  $G_0 \triangleleft G$ .

**Definition 5.2.** We say that  $G$  is *isotopically factorizable* if for every open cover  $\mathcal{U}$  and every isotopy  $f \in \mathcal{P}G$  there are  $U_1, \dots, U_r \in \mathcal{U}$  and  $f_1, \dots, f_r \in \mathcal{P}G$  such that  $f = f_1 \dots f_r$  and  $\text{supp}(f_i) \subset U_i$  for all  $i$ .

Clearly, if  $G$  is isotopically factorizable then  $G_0$  is factorizable.

**Proof of Theorem 1.4.** For  $f \in \mathcal{P}G$  by  $\langle f \rangle_\sim$  denote the homotopy rel. endpoints class of  $f$ .

Due to Proposition 3.2 and the assumption, for any  $x \in X$  there is  $g, \bar{g} \in [G_0, G_0]$  such that  $[g, \bar{g}](x) \neq x$ . Consequently, there exists an open cover  $\mathcal{U}$  such that for all  $U \in \mathcal{U}$  there are  $g, \bar{g} \in [G_0, G_0]$  such that  $[g, \bar{g}](U) \cap U = \emptyset$ . Since  $G_0 \triangleleft G$ , the same holds for  $\mathcal{U}^G$  instead of  $\mathcal{U}$ . In view of Lemma 5.1, there are  $f, \bar{f} \in \mathcal{P}^\square G$  such that  $f_1 = g$  and  $\bar{f}_1 = \bar{g}$ .

Choose  $\mathcal{V}$  such that  $\mathcal{V} \prec \mathcal{U}$  (Lemma 3.3) and denote

$$\mathcal{P}G^\mathcal{U} = \prod_{U \in \mathcal{U}^G} [\mathcal{P}G_U, \mathcal{P}G_U].$$

First we notice that  $[\mathcal{P}G, \mathcal{P}G] \subset \mathcal{P}G^\mathcal{U}$ . As in the proof of Theorem 1.3 we use (3.1) for elements of  $\mathcal{P}G$  and the fact that  $\mathcal{P}G$  is  $\mathcal{V}$ -factorizable.

Next, fix arbitrarily  $U \in \mathcal{U}$  and let  $f, \bar{f} \in \mathcal{P}^\square G$  as above. Put  $\hat{f} = [f, \bar{f}]$ . Then  $\hat{f}_t(U) \cap U = \emptyset$  for all  $t \in [\frac{1}{2}, 1]$ . We will show that for every  $h, \bar{h} \in \mathcal{P}G_U$  the bracket  $[\langle h \rangle_\sim, \langle \bar{h} \rangle_\sim]$  is represented as a product of four commutators of elements of  $[\tilde{G}, \tilde{G}]$ . In view of Lemma 5.1 choose  $k, \bar{k} \in \mathcal{P}^*G$  such that  $\langle k \rangle_\sim = \langle h \rangle_\sim$  and  $\langle \bar{k} \rangle_\sim = \langle \bar{h} \rangle_\sim$ . It follows that  $[\hat{f}k\hat{f}^{-1}, \bar{k}] = \text{id}$  and  $[[k, \hat{f}], \bar{k}] = [k, \bar{k}]$ . Therefore,  $[\langle h \rangle_\sim, \langle \bar{h} \rangle_\sim]$  is a product of four commutators of elements of  $[\tilde{G}, \tilde{G}]$ .  $\square$

**Remark 5.3.** (1) Observe that one can formulate some results for  $[\tilde{G}, \tilde{G}]$ , analogous to Theorems 1.3, 3.5 and 3.8, by assuming that  $G$  is isotopically factorizable,  $G_0$  satisfies some conditions in Definition 3.1,  $\text{cl}_{\mathcal{P}G}$  is bounded, and  $\mathcal{P}G$  is bounded in  $\text{frag}^\mathcal{U}$ .

(2) Obviously,  $\tilde{G}$  and  $[\tilde{G}, \tilde{G}]$  are not simple, since  $\pi(G) \triangleleft \tilde{G}$  and  $[\pi(G), \pi(G)] \triangleleft [\tilde{G}, \tilde{G}]$ , where  $\pi(G)$  is the fundamental group of  $G$ .

## 6. The commutator subgroup of a diffeomorphism group on an open manifold

Assume  $r = 0, 1, \dots, \infty$ . Let a manifold  $M$  be the interior of a compact, connected manifold  $\bar{M}$  of class  $C^r$  with non-empty boundary  $\partial$ . By a *product neighborhood* of  $\partial$  we mean a closed subset  $P = \partial \times [0, 1)$  of  $M$  such that  $\partial \times [0, 1]$  is embedded in  $\bar{M}$ , and  $\partial \times \{1\}$  is identified with  $\partial$ .

A *translation system* on the product manifold  $N \times [0, \infty)$  (cf. [14, p. 168]) is a family  $\{P_j\}_{j=1}^\infty$  of closed product neighborhoods of  $N \times \{\infty\}$  such that  $P_{j+1} \subset \text{int } P_j$  and  $\bigcap_{j=1}^\infty P_j = \emptyset$ . By a *ball* we mean an open ball with its closure compact and contained in a chart domain.

Let  $G \leq \mathcal{D}^r(M)$ , where  $r = 0, 1, \dots, \infty$ . For a subset  $U \subset M$  denote by  $G(U)$  the subgroup of all elements of  $G$  which can be joined with the identity by an isotopy in  $G$  compactly supported in  $U$ .

**Definition 6.1.** Let  $\mathcal{B}$  be a cover of  $M$  by balls.  $G$  is called  *$\mathcal{B}$ -factorizable* if for any  $f \in G$  there are a product neighborhood  $P = \partial \times [0, 1)$ , and a family of diffeomorphisms  $g, g_1, \dots, g_\rho \in G$  such that:

(1)  $f = gg_1 \dots g_\rho$  with  $g \in G(P)$  and  $g_j \in G(B_j)$ , where  $B_j \in \mathcal{B}$  for  $j = 1, \dots, \rho$ .

Furthermore, for any product neighborhood  $P$  and for any  $g \in G(P)$  there is a sequence of reals from  $(0, 1)$  tending to 1

$$0 < a_1 < \bar{a}_1 < \bar{b}_1 < b_1 < a_2 < \dots < a_n < \bar{a}_n < \bar{b}_n < b_n < \dots < 1$$

and  $h \in G(P)$  such that:

(2)  $h = g$  on  $\bigcup_{n=1}^\infty \partial \times [\bar{a}_n, \bar{b}_n]$ ;

(3)  $h = \text{id}$  if  $g = \text{id}$ .

Put  $D_n := \partial \times (a_n, b_n)$  and  $D := \bigcup_{n=1}^\infty D_n$ . Then we also assume that:

(4)  $\text{supp}(h) \subset D$ ;

(5) for the resulting decomposition  $h = h_1 h_2 \dots$  with respect to  $D = \bigcup_{n=1}^\infty D_n$  we have  $h_n \in G(D_n)$  for all  $n$ .

$G$  is called *factorizable (in the wider sense)* if it is  $\mathcal{B}$ -factorizable for every cover  $\mathcal{B}$  of  $M$  by balls.

Finally, if  $G$  factorizable, for any  $f \in G$  we define  $\text{Frag}_G(f)$  as the smallest  $\rho$  such that there are a family of balls  $\{B_j\}$ , a product neighborhood  $P$  and a decomposition of  $f$  as in (1). Then  $\text{Frag}_G$  is a conjugation-invariant norm on  $G$ , called the *fragmentation norm*. In fact, since  $G \leq \mathcal{D}^r(M)$ , any  $g \in G$  does not change the ends of  $M$  so that it takes (by conjugation) any decomposition as in (1) into another such a decomposition.

Define  $\text{Frag}_G := \sup_{g \in G} \text{Frag}_G(g)$ , the diameter of  $G$  in  $\text{Frag}_G$ . Consequently,  $\text{Frag}_G$  is bounded iff  $\text{Frag}_G < \infty$ .

**Remark 6.2.** The reason for introducing Definition 6.1 is the absence of isotopy extension theorems or fragmentation theorems for some geometric structures. Roughly speaking,  $G$  satisfies Definition 6.1 if all its elements can be joined with id by an isotopy in  $G$  and appropriate versions of the above mentioned theorems are available.

Let  $\text{Diff}^r(M)$  (resp.  $\text{Diff}_c^r(M)$ ) be the group of all  $C^r$  diffeomorphisms of  $M$  (resp. with compact support). To illustrate Definition 6.1 we consider the following:

**Example 6.3.** The group  $\text{Diff}^r(\mathbb{R}^n)$  does not satisfy Definition 6.1. The reason is that in this case any  $f \in \text{Diff}^r(\mathbb{R}^n)$  would be isotopic to id due to 6.1(1) which is not true. Next, any  $f \in \text{Diff}_c^r(\mathbb{R}^n)$  is isotopic to the identity but the isotopy need not be compactly supported. It follows that  $\text{Diff}_c^r(\mathbb{R}^n)$  does not fulfill Definition 6.1(1). The exception is  $r = 0$ , when the Alexander trick is in use (see e.g. [7, p. 70]) and any compactly supported homeomorphism on  $\mathbb{R}^n$  is isotopic to id by a compactly supported isotopy. It follows that  $\text{Diff}_c^0(\mathbb{R}^n)$  is factorizable in view of [7].

Let  $C = \mathbb{R} \times \mathbb{S}^1$  be the annulus. Then there is the twisting number epimorphism  $\text{Diff}_c^r(C) \rightarrow \mathbb{Z}$ . It follows that  $\text{Diff}_c^r(C)$  is unbounded in view of Lemma 1.10 in [5]. On the other hand,  $\text{Diff}_c^r(C)$  is not factorizable.

#### Definition 6.4.

- (1)  $G$  is said to be *determined on compact subsets* if the following is satisfied. Let  $f \in \mathcal{D}^r(M)$ . If there are a sequence of relatively compact subsets  $U_1 \subset \bar{U}_1 \subset U_2 \subset \cdots \subset U_n \subset \bar{U}_n \subset U_{n+1} \subset \cdots$  with  $\bigcup U_n = M$  and a sequence  $\{g_n\}$ ,  $n = 1, 2, \dots$ , of elements of  $G$  such that  $f|_{U_n} = g_n|_{U_n}$  for  $n = 1, 2, \dots$ , then we have  $f \in G$ .
- (2) We say that  $G$  *admits translation systems* if for any sequence  $\{\lambda_n\}$ ,  $n = 0, 1, \dots$ , with  $\lambda_n \in (0, 1)$ , tending increasingly to 1, there exists a  $C^r$ -mapping  $[0, \infty) \ni t \mapsto f_t \in G$  supported in the interior of  $P$ , with  $f_0 = \text{id}$ ,  $f_j = (f_1)^j$  for  $j = 2, 3, \dots$ , and such that for the translation system  $P_n = \partial_i \times [\lambda_n, 1)$  one has  $f_1(P_n) = P_{n+1}$  for  $n = 0, 1, 2, \dots$ .

By using suitable isotopy extension theorems (cf. [7,12,4]) we have:

**Proposition 6.5.** ([24]) The groups  $\mathcal{D}^r(M)$ ,  $r = 0, 1, \dots, \infty$ , satisfy Definitions 6.1 and 6.4.

The following result is essential to describe the structure of  $[G, G]$ . Though it was proved in [24], we give the proof of it for the sake of completeness.

**Lemma 6.6.** If  $G$  satisfies Definitions 6.1 and 6.4, then any  $g \in G(P)$ , where  $P$  is a product neighborhood of  $\partial$ , can be written as a product of two commutators of elements of  $G(P)$ .

**Proof.** We may assume that  $g \in G(\text{int}(P))$ . Choose as in Definition 6.1 a sequence  $0 < a_1 < \bar{a}_1 < b_1 < a_2 < \cdots < a_n < \bar{a}_n < b_n < \cdots < 1$  and  $h \in G(P)$  such that conditions (2)–(5) in Definition 6.1 are fulfilled. Put  $\tilde{h} = h^{-1}g$ , that is  $g = h\tilde{h}$ . Then  $\text{supp}(\tilde{h})$  is in  $(0, \bar{a}_1) \cup \bigcup_{n=1}^{\infty} (\bar{b}_n, \bar{a}_{n+1})$ , and  $\tilde{h} = g$  on  $[0, a_1] \cup \bigcup_{n=1}^{\infty} [b_n, a_{n+1}]$ . We show that  $h$  is a commutator of elements in  $G(\text{int}(P))$ .

Choose arbitrarily  $\lambda_0 \in (0, a_1)$  and  $\lambda_n \in (b_n, a_{n+1})$  for  $n = 1, 2, \dots$ . In light of Definition 6.4(2) there exists an isotopy  $[0, \infty) \ni t \mapsto f_t \in G$  supported in  $\partial \times (0, 1)$ , such that  $f_0 = \text{id}$  and  $f_j(P_n) = P_{n+j}$  for  $j = 1, 2, \dots$  and for  $n = 0, 1, 2, \dots$ , where  $P_n = \partial \times [\lambda_n, 1)$  for  $n = 0, 1, \dots$ . Now define  $\tilde{h} \in G(\text{int}(P))$  as follows. Set  $\tilde{h} = h$  on  $\partial \times [0, \lambda_1)$ , and  $\tilde{h} = h(f_1 h f_1^{-1}) \cdots (f_n h f_n^{-1})$  on  $\partial \times [0, \lambda_{n+1})$  for  $n = 1, 2, \dots$ . Here  $f_n = (f_1)^n$ . Then  $\tilde{h}|_{\partial \times [0, \lambda_n)}$  is a consistent family of functions, and  $\tilde{h} = \bigcup_{n=1}^{\infty} \tilde{h}|_{\partial \times [0, \lambda_n)}$  is a local diffeomorphism. It is easily checked that  $\tilde{h}$  is a bijection. Due to Definition 6.4(1)  $\tilde{h} \in G(\text{int}(P))$ .

By definition we have the equality  $\tilde{h} = h f_1 \tilde{h} f_1^{-1}$ . It follows that  $h = \tilde{h} f_1 \tilde{h}^{-1} f_1^{-1} = [\tilde{h}, f_1]$ . Similarly,  $\tilde{h}$  is a commutator of elements of  $G(P)$ . The claim follows.  $\square$

**Definition 6.7.** Let  $G$  satisfy Definition 6.1. Then

- (1) the symbol  $G_c$  stands for the subgroup of all  $f \in G$  such that there is a decomposition  $f = g g_1 \cdots g_\rho$  as in Definition 6.1(1) with  $g = \text{id}$ ;
- (2)  $G$  is said to be *localizable* if for any  $f \in G$  and any compact  $C \subset M$  there is  $g \in G_c$  such that  $f = g$  on  $C$ .

Clearly  $G_c$  is a subgroup of the group of compactly supported members of  $G$ . However, the converse is not true: for  $G = \mathcal{D}^r(C)$  take a compactly supported diffeomorphism of  $C$  with nonzero twisting number (Example 6.3). For the reason of introducing localizable groups, see Remark 6.2. It follows from the isotopy extension theorems (see [7,12]) that  $\mathcal{D}^r(M)$  is localizable.

**Proposition 6.8.** Let  $\text{frag}_G = \text{frag}_G^{\mathcal{B}}$ , where  $\mathcal{B}$  the family of all balls on  $M$  (cf. Section 2). We have  $\text{frag}_{G_c} = \text{Fragd}_{G_c}$ .



**Proof.** If  $g \in G_c$  then  $\text{Frag}_{G_c}(g) \leq \text{frag}_{G_c}(g)$ , since any fragmentation of  $g$  supported in balls is of the form from Definition 6.1(1). On the other hand, if  $g = g_0 g_1 \dots g_{\rho'}$  with  $\rho' < \rho = \text{frag}_{G_c}(g)$  is as in 6.1(1), then  $g_0^{-1}g \in G_c$  and  $\text{frag}_{G_c}(g_0^{-1}g) \leq \rho'$ . Thus,  $\text{fragd}_{G_c} = \text{Fragd}_{G_c}$ .  $\square$

For any  $M$  as above a theorem of McDuff [17] states that  $\mathcal{D}^r(M)$  is perfect. We generalize it as follows.

**Theorem 6.9.** *Let  $M$  be an open  $C^r$ -manifold ( $r = 0, 1, \dots, \infty$ ) such that  $M = \text{int } \bar{M}$ , where  $\bar{M}$  is a compact manifold. Suppose that  $G \leq \mathcal{D}^r(M)$  satisfies Definitions 6.1, 6.4 and 6.7, and that  $G_c$  is non-fixing. Then  $[G, G]$  is perfect.*

**Proof.** In view of Definition 6.1 for an arbitrary  $f \in G$  we can write  $f = gh$ , where  $g \in G(P)$  and  $h \in G_c$ . Let  $[f_1, f_2] \in [G, G]$  with  $f_1 = g_1 h_1$  and  $f_2 = g_2 h_2$  as above. Since  $G_c$  is localizable we have  $[g_1, h_2], [g_2, h_1] \in [G_c, G_c]$ . Due to Lemma 6.6  $G(P)$  is perfect, that is  $g_1, g_2 \in [G, G]$ . It follows from (3.1) that  $[f_1, f_2] = \varphi[k_1, k_2][k'_1, k'_2][k''_1, k''_2]$ , where  $\varphi \in [[G, G], [G, G]]$  and  $k_1, k_2, k'_1, k'_2, k''_1, k''_2 \in G_c$ . But by Theorem 1.2  $[G_c, G_c]$  is also perfect. It follows that  $[G, G]$  is perfect too.  $\square$

**Theorem 6.10.** *Under the assumptions of Theorem 6.9, if  $\text{cl}_{G_c}$  and  $\text{frag}_G^{\mathcal{V}}$  are bounded, where  $\mathcal{V}$  is an arbitrary open cover with  $\mathcal{V} \prec \mathcal{B}$ , then  $[G, G]$  is uniformly perfect.*

**Proof.** By Theorem 6.9,  $[G, G]$  is perfect. In view of Proposition 3.2,  $[G, G]$  is 1-non-fixing. Due to this fact and Lemma 3.3 we can find an open cover  $\mathcal{U}$  such that  $\mathcal{U} \prec \mathcal{B}$  and such that for each  $U \in \mathcal{U}$  there are  $h_1, h_2 \in [G, G]$  with  $U \cap [h_1, h_2](U) = \emptyset$ . We denote

$$G^{\mathcal{U}} = \prod_{U \in \mathcal{U}} [G_U, G_U].$$

Here  $\mathcal{U}^G := \{g(U) : g \in G \text{ and } U \in \mathcal{U}\}$ . Then also for each  $U \in \mathcal{U}^G$  there is  $h_1, h_2 \in [G, G]$  with  $U \cap [h_1, h_2](U) = \emptyset$ .

Assume that  $\mathcal{V} \prec \mathcal{U}$  and  $\text{frag}_G^{\mathcal{V}} = \rho$ . Let  $[f_1, f_2] \in [G, G]$ . As in the proof of Theorem 6.9 we have

$$[f_1, f_2] = [g_1, g_2][h_1, h_2][h'_1, h'_2][h''_1, h''_2],$$

where  $g_1, g_2 \in G(P)$  and  $h_1, \dots, h''_2 \in G_c$ . By Lemma 6.6 and (3.1),  $[g_1, g_2]$  is a product of four commutators of elements of  $[G, G]$ .

Next, any  $[h_1, h_2] \in [G_c, G_c]$  can be expressed as a product of at most  $\rho^2$  elements of  $G^{\mathcal{U}}$  of the form  $[k_1, k_2]$ , where  $k_1, k_2 \in G_U$  for some  $U$ . In fact, it is a consequence of (3.1) and the fact that  $\mathcal{V} \prec \mathcal{U}$ . Now if  $\text{cld}_{G_c} = d$ , then every element of  $[G_c, G_c]$  is a product of at most  $d\rho^2$  elements of  $G^{\mathcal{U}}$  of the form  $[k_1, k_2]$ , where  $k_1, k_2 \in G_U$  for some  $U$ .

Finally, fix arbitrarily  $U \in \mathcal{U}^G$ . We wish to show that for every  $k_1, k_2 \in G_U$  the bracket  $[k_1, k_2]$  can be represented as a product of four commutators of elements of  $[G, G]$ . By assumption on  $\mathcal{U}^G$ , there are  $h_1, h_2 \in [G, G]$  such that  $h(U) \cap U = \emptyset$  for  $h = [h_1, h_2]$ . It follows that  $[hk_1 h^{-1}, k_2] = \text{id}$ . Therefore,  $[[k_1, h], k_2] = [k_1, k_2]$ . Observe that indeed  $[[k_1, h], k_2]$  is a product of four commutators of elements of  $[G, G]$ . Thus any element of  $[G, G]$  is a product of at most  $4d(1 + \rho^2)$  commutators of elements of  $[G, G]$ .  $\square$

**Corollary 6.11.** *Suppose that the assumptions of Theorem 6.9 are fulfilled and that  $G$  is bounded. Then  $[G, G]$  is uniformly perfect.*

In fact,  $\text{cl}_G$  is bounded in view of Proposition 1.4 in [5], and  $\text{frag}_{G_c}$  is bounded in view of Proposition 6.8.

**Remark 6.12.** By using Theorems 3.5 and 3.8, Lemma 6.6 and (3.1) we can obtain some estimates on  $\text{cl}_{[G, G]}$ .

## 7. Examples and open problems

Let  $M$  be a paracompact manifold, possibly with boundary, of class  $C^r$ ,  $r = 0, 1, \dots, \infty$ .

1. Let  $M$  be a manifold with a boundary,  $\dim(M) = n \geq 2$ . Then  $G = \mathcal{D}_c^r(M)$ , where  $r = 0, 1, \dots, \infty$ ,  $r \neq n$  and  $r \neq n + 1$  is perfect (see [20,19]) and non-simple. Recently, Abe and Fukui [2], using results of Tsuboi [29] and their own methods, showed that  $G$  is also uniformly perfect for many types of  $M$ . In the remaining cases, where we do not know whether  $G$  is perfect or uniformly perfect, our results are of use.

2. Let  $N$  be a submanifold of  $M$  of class  $C^r$ ,  $r = 0, 1, \dots, \infty$ , and  $\dim N \geq 1$ . It was proved in [21] that  $G_c$ , where  $G = \mathcal{D}^r(M, N)$  is the identity component of the group of  $C^r$ -diffeomorphisms preserving  $N$ , is perfect. The same was proved in the Lipschitz category in [1]. All these groups are clearly non-simple. It follows from [2] that  $G_c$  is also uniformly perfect for many types of pairs  $(M, N)$ . Several results of the present paper give new information on the structure of  $G$  and  $G_c$ .

3. Given a foliation  $\mathcal{F}$  of dimension  $k$  on a manifold  $M$ , let  $G = \mathcal{D}^r(M, \mathcal{F})$  be the identity component group of all diffeomorphisms of class  $C^r$  taking each leaf to itself. Due to results of Rybicki [18], Fukui and Imanishi [9] and Tsuboi [28], the group  $G_c$  is perfect provided  $r = 0, 1, \dots, k$  or  $r = \infty$ . It is very likely that for large (but finite)  $r$  the group  $\mathcal{D}_c^r(M, \mathcal{F})$

is not perfect (cf. a discussion on this problem in [13]). It is a highly non-trivial problem whether  $G_c$  is uniformly perfect. Several results of the present paper apply to  $G_c$  or  $G$ .

4. Let  $\mathcal{F}$  be a foliation of dimension  $k$  on the Lipschitz manifold  $M$  and let  $G = \text{Lip}(M, \mathcal{F})$  be the group of all Lipschitz homeomorphisms taking each leaf of  $\mathcal{F}$  to itself. In view of results of Fukui and Imanishi [10], the group  $G_c$  is perfect. Further results may be concluded from our paper.

5. Assume now that  $\mathcal{F}$  is a singular foliation, i.e. the dimensions of its leaves need not be equal (see [26]). One can consider the group of leaf-preserving diffeomorphisms of  $\mathcal{F}$ ,  $G = \mathcal{D}^\infty(M, \mathcal{F})$ . However, it is hopeless to obtain any perfectness results for this group. On the other hand, Theorem 1.2 still works in this case and we know that the commutator group  $[G_c, G_c]$  is perfect. We do not know whether  $[G_c, G_c]$  is uniformly perfect.

6. Let us recall the definition of Jacobi manifold (see [6]). Let  $M$  be a  $C^\infty$  manifold, let  $\mathfrak{X}(M)$  be the Lie algebra of the vector fields on  $M$  and denote by  $C^\infty(M, \mathbb{R})$  the algebra of  $C^\infty$  real-valued functions on  $M$ . A *Jacobi structure* on  $M$  is a pair  $(\Lambda, E)$ , where  $\Lambda$  is a 2-vector field and  $E$  is a vector field on  $M$  satisfying

$$[\Lambda, \Lambda] = 2E \wedge \Lambda, \quad [E, \Lambda] = 0.$$

Here,  $[\cdot, \cdot]$  is the Schouten–Nijenhuis bracket. The manifold  $M$  endowed with the Jacobi structure is called a *Jacobi manifold*. If  $E = 0$  then  $(M, \Lambda)$  is a Poisson manifold. Observe that the notion of Jacobi manifold generalizes also symplectic, locally conformal symplectic and contact manifolds.

Now, let  $(M, \Lambda, E)$  be a Jacobi manifold. A diffeomorphism  $f$  on  $M$  is called a *hamiltonian diffeomorphism* if, by definition, there exists a hamiltonian isotopy  $f_t$ ,  $t \in [0, 1]$ , such that  $f_0 = \text{id}$  and  $f_1 = f$ . An isotopy  $f_t$  is *hamiltonian* if the corresponding time-dependent vector field  $X_t = \dot{f}_t \circ f_t^{-1}$  is hamiltonian.

Let  $G = \mathcal{H}(M, \Lambda, E)$  be the compactly supported identity component of all hamiltonian diffeomorphisms of class  $C^\infty$  of  $(M, \Lambda, E)$ . It is not known whether  $G$  is perfect, even in the case of regular Poisson manifold (see [22]). However, by Theorem 1.2 the commutator group  $[G, G]$  is perfect. It is an interesting and difficult problem to answer when  $[G, G]$  is uniformly perfect.

In the transitive cases, the compactly supported identity components of the hamiltonian symplectomorphism group and the contactomorphism group are simple (see [3,11,23]). In general,  $G$  and  $\bar{G}$  is not uniformly perfect in the symplectic case, see [5]. An obstacle for the uniform simplicity of the first group is condition (2) in Theorem 4.3. On the other hand, the contactomorphism group satisfies this condition and it is likely that for some contact manifolds it is uniformly simple.

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